Matrix Transformations between the Sequence Spaces of Maddox

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Abstract
In this research work we characterized the matrix classes $(l(p), V_\sigma^\infty)$ *and* $(l_\infty(p), V_\sigma^\infty)$ *and determine the necessary and sufficient conditions on the matrix sequence* $A = (A_n)$ *in order that* A *belongs to the matrix classes* $(l(p), V_\sigma^\infty)$ *and* $(l_\infty(p), V_\sigma^\infty)$.

Keywords: Sequence spaces, matrix transformation, necessary conditions, sufficient conditions.

I. Introduction

A sequence space is a vector space or a linear space whose elements are infinite sequences of real or complex numbers. Equivalently, it is a function space whose elements are functions from the natural number to the field K of real or complex numbers.

The classical sequence spaces are expressed as follows:

$$l_\infty = \{x = (x_k) \in \omega : sup_k |x_k| < \infty\}$$

$$c = \left\{x = (x_k) \in \omega : \lim_{k\to\infty} x_k = l \, for \; some \; l\right\}$$

$$c_0 = \left\{x = (x_k) \in \omega : \lim_{k\to\infty} x_k = 0\right\}$$

It is not difficult to see that l_∞, c *and* c_0 are normed linear spaces such that $c_0 \subset c \subset l_\infty$.

Definition: An invariant (or fixed) point is one which is mapped onto itself: that is, it is its own image.

Definition: Let σ be a one-one mapping from the set $\mathbb{N}$ of natural numbers into itself. A continuous linear functional $\emptyset$ on the space l_∞ is said to be an invariant mean or $\sigma - mean$ if and only if

(i) $\emptyset(x) \geq 0,\ when\ the\ sequence\ x = (x_k)\ has\ x_k \geq 0\ \ for\ all\ k$
(ii) $\emptyset(e) = 1\ where\ e = (1, 1, 1, \dots)\ and$
(iii) $\emptyset(x) = \emptyset(x_{\sigma(k)})\ for\ all\ x \, \epsilon l_\infty$.(Mursaleen, 2012)

Definition: Under translation each point is moved a fixed distance in a given direction.

If σ is the translation mapping $n \to n+1,\ then\ \sigma - mean$ is often called a Banach limit and the set W_σ of bounded sequences all of whose invariant means are equal reduces to the set f of almost convergence sequences studied by Aiyub(2012).

Note that $\sigma - mean$ extends the limit functional on c in the sense that $\emptyset(x) = limx$ for all $x \in c$ if and only if σ has no finite orbit that is to say, if and only if for all $n \geq 0, m \geq 1, \sigma^m(n) \neq n$. Ali Fares (2020)

Definition: A bounded sequence $x = (x_k)$ is said to be $\sigma - convergent\ if\ and\ only\ if\ x\epsilon W_\sigma$ such that $\sigma^m(n) \neq n\ for\ all\ n \geq 0, m \geq 1$ Hakan(2014)

$$V_\sigma = \left\{x\epsilon l_\infty : \lim_m t_{mn}(x) = L\ uniformly\ \ in\ n\right\} \text{ where}$$

$$L = \sigma - limx \text{ and}$$

$$t_{mn}(x) = (m+1)^{-1}[x_n + Tx_n + T^2x_n + \cdots + T^m x_n],\ t_{-1} = 0$$

When pk is real such that pk>0 and sup pk<∞, we define

$$l_\infty(p) = \{x = (x_k) : sup|x_k|^{pk} < \infty\}$$

$$l(p) = \left\{x = (x_k) : \sum |x_k|^{pk} < \infty\right\}$$

$$c(p) = \{x = (x_k) : |x_k - l|^{pk} \to 0\ \ for\ some\ \ l\}$$ Mohiuddine(2010).

Definition: A matrix is a rectangular array of numbers. In other words, matrix A is an object acting on X by multiplication to produce a new vector Ax. Each entry in the matrix is called an element. Matrices are classified by the number of rows and the number of columns that they have.

Definition: Let X be a linear space and d a metric on X. Then (X, d) is said to be a linear metric space , if the algebraic operations on X are continuous functions.

II. Statement of the Problem

A transformation T from $R^n\ to\ R^m$ is a rule that assigns to each vector X in R^n a vector T(X) in R^m.

$$T: R^n \to R^m$$

$R^n: domain\ of\ T,\ \ R^m: codomain\ of\ T,\ T(X)\ in\ R^m\ is\ the\ image\ of\ X$ under the transformation T.

The set of all images is the range of T.

Suppose $A = (a_{nk})$ is an infinite matrix of real numbers a_{nk} , $where\ n, k \in \mathbb{N}$. Then we obtain the sequence $(A_n x)$, the A-transform of x by the usual matrix product.

$$Ax = \begin{bmatrix} a_{11} & a_{12} \dots & a_{1k} \dots \\ a_{21} & a_{22} \dots & a_{2k} \dots \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nk} \dots \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \\ \vdots \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + & a_{12}x_2 + & a_{13}x_3 + \cdots \\ a_{21}x_1 + & a_{22}x_2 + & a_{23}x_3 + \cdots \\ \vdots & \vdots & \vdots \\ a_{n1}x_1 + & a_{n2}x_2 + & a_{n3}x_3 + \cdots \\ & \vdots & \end{bmatrix}$$

Hence in this way, we transform the sequence x into the sequence $Ax = \{(A_n x)\}\ \ with$

$$(A_n x) = \sum_k a_{nk} x_k$$

Provided the series on the right hand side converges for some n, that is to say the limit of the sequence of partial sum converges partially with respect to the induced metric.

III. Significance of the Study

One can ask why we employ the special transformations represented by linear operators. The answer to this question is that, in many cases, the most general linear operators between two sequence spaces are given by an infinite matrix. So the theory of matrix transformation has always been of great interest in the study of sequence spaces. The theory of infinite matrices and sequence spaces has been developed and studied by Richard Cooke G. (1950).

IV. Literature Review

The approach of constructing a new sequence spaces by means of matrix transformations of a particular limitation method has been studied by several authors ; Schaefer [1972] defined the concepts of $\sigma - conservative$, $\sigma - regular\ and\ \sigma - coercive\ matrices$ and characterized the matrix classes (c, V_σ) , $(C,\ V_\sigma)_{reg}\ and\ (l_\infty, V_\sigma)$ where V_σ denotes the set of all bounded sequences all of whose invariant means are equal . Mursaleen[2009] characterized the classes $(c(p), V_\sigma)$, $(c(p), V_\sigma)_{reg}\ and\ (l_\infty(p), V_\sigma)$ of matrix, which generalized the result of Schaefer [1972]. Mohiuddine and Aiyub[2012] defined the space $\omega(p, s)$ and obtained necessary and sufficient conditions to characterize the matrix of classes $(\omega(p, s), V_\sigma)$, $\left(\omega_p(s), V_\sigma\right)\ and\ \left(\omega_p(s), V_\sigma\right)_{reg}$.

The sequence space V_σ was introduced and studied by Schaefer (1972). In the present research work we extend $V_\sigma\ to\ V_\sigma^\infty$ which is related to the concept of invariant mean $(\sigma - mean)$ and characterized the matrix classes $(l(p), V_\sigma^\infty)$ and $(l_\infty(p), V_\sigma^\infty)$. Further, we also determine the necessary and sufficient conditions on the matrix sequence $A = (A_n)$ in order that A belongs to the matrix classes $(l(p), V_\sigma^\infty)$ and $(l_\infty(p), V_\sigma^\infty)$ where $l(p)\ and\ l_\infty(p)$ are the sequence spaces of Maddox(1967) with sequence $p = (p_k)$ as parameter, generalize the classical sequence spaces $l_1\ and\ l_\infty$.

Let ω be the set of all complex sequences $x = (x_k)_{k=0}^\infty$. Let φ , l_∞ , $c\ \ and\ \ c_0$ denote the sets of all finite, bounded, convergent and null sequences respectively. We write

$l_p = \{x \in \omega : \sum_{k=0}^{\infty}|x_k| < \infty\}$ for $1 \le p < \infty$. By e and $e^{(n)}$ $(n \in \mathbb{N})$.

We denote the sequences such that $e_k = 1$ for $k = 0, 1 \ldots$

Note that c_0, c and l_∞ are Banach spaces with the sup-norm $\|x\|_\infty = sup_k|x_k|$ and $l^p(1 \le p < \infty)$ are Banach spaces with the norm $\|x\|_p = (\sum|x_k|^p)^{\frac{1}{p}}$ while φ is not a Banach space with respect to any norm Mursaleen and Mohiuddine (2010)

V. Methodology

We employed the method of summability and operator theory for the characterization of classes of matrix transformations between sequence spaces which constitute a wide, interesting and important field in mathematics . These results are needed to determine the corresponding subclasses of compact matrix operators and more recently of general linear operators between the respective sequence and solvability of infinite system of linear equations .

VI. Main Results

Let X and Y be two sequence spaces and $A = (a_{nk})_{n;k=1}^{\infty}$ be an infinite matrix of real or complex numbers. We write $Ax = \big(A_n(x)\big)$ $where$ $A_n(x) = \sum_k a_{nk}x_k$ provided that the series on the right converges for some n.

If $x = (x_k) \in X$ $implies$ $that$ $Ax \in Y$, then we say that A defines a matrix transformation from X into Y and we denote the class of such matrices by (X, Y). Since Ax is defined, then for all $n, m \ge 0$

$$t_{mn}(Ax) = \sum_{k=1}^{\infty} t(n,k,m)x_k \quad where$$

$t(n,k,m) = \frac{1}{m+1}\sum_{j=0}^{\infty} a\big(\sigma^j(n),k\big)$ and $a(n,k)$ denotes the element a_{nk} of the matrix A.

Note that if σ is a translation, then V_σ^∞ is reduced to the space $f_\infty = \{x \in l_\infty : sup_{m,n}|g_{mn}(x)| < \infty\}$ where $g_{mn}(x) = \frac{1}{m+1}\sum_{k=0}^{\infty} x_{k+n}$ Mursaleen (1978) We called the space V_σ^∞ as the space of $\sigma -$ $bounded$ $sequences$. It is clear that $c \subset V_\sigma \subset V_\sigma^\infty \subset l_\infty$ Aiyub (2012) . V_σ^∞ is Banach space normed by

$$\|x\| = sup_{n,m}|t_{mn}x| \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Theorem 1.1

Show that the matrix A belongs to $(l(p), V_\sigma^\infty)$ if and only if there exists $B \ge 1$ such that for

$sup_{m,n}|t(n,k,m)|^{qk}B^{-qk} < \infty \quad (1 < pk < \infty)$

$sup_{m,n,k}|t(n,k,m)|^{pk} < \infty \quad (0 < pk \leq 1)$ …………………………………… (2)

Proof

We consider the case $1 < pk < \sup pk < \infty, \quad for\ all\ k.$

Necessary condition : $A \in (l(p), V_\sigma^\infty)\ and\ x \in l(p).\ We\ put\ g_n(x) = sup_m \sum_k |t(n,k,m)x_k|^{pk}$ Then it is easy to see that for $n \geq 0$, g_n is a continuous seminorm on $l_\infty\ and\ g_n$ is a pointwise bounded on $l(p)$. Suppose that (2) does not hold. Then on the sequence space related to invariant mean there exist $x \in l(p)$ with $sup_n g_n(x) = \infty$. By the principal of condensation of singularities; the set $\{x \in l(p): sup_n g_n(x) = \infty\}$ is of second category in $l(p)$ and hence there exist $x \in l(p)$ with $sup_n g_n(x) = \infty$. But this contradicts that (g_n) is pointwise bounded on $l(p)$. Now by the Banach-Steinhauss theorem, there is a constant M such that

$g_n(x) \leq M\|x\|$ ……………………………………………… (3)

Applying equation (3) to the sequence $x = (x_k)$ defined by Schaefer, by replacing (a_{nk}) with $t(n,k,m)$, we obtained the necessary of (2)

Sufficient conditions:

Let (2) hold and $x \in l(p)$.

Using the inequalities $|ab| \leq C(|a|^q C^{-q} + |b|^p)$, C > 0.

We have for some integer B > 1

$\sum_k |t(n,k,m)x_k| \leq B(\sum_k |t(n,k,m)|B^{-qk} + |x_k|^{pk}) \quad for\ every\ x \in l(p).$

Therefore by (2) $sup_{n,m} \sum_k |t(n,k,m)x_k| < \infty$.

That is $Ax \in V_\sigma^\infty\ for\ x \in l(p).$

Hence $A \in (l(p), V_\sigma^\infty)$.

Threorem 1.2

$A \in (l(p), f_\infty)$ if and only if there exists an integer N > 1 such that

$sup_{n,m}\left\{\sum_k |t(n,k,m)|^{q_k} N^{1/q_k}\right\} < \infty$, $(1 < p_k < \infty, \frac{1}{p_k} + \frac{1}{q_k} = 1)$ …… (I)

$sup_{n,m}|t(n,k,m)|^{p_k} < \infty$, $(0 < p_k \leq 1)$. ……………………………… …… (ii)

Proof

Necessity: suppose $A \in (l(p), f_\infty)\ \ and\ put\ T_{n,m}(x) = \psi_{n,m}(Ax)\ with\ T_n(x) = sup_m|\psi_{n,m}(Ax)|$

We see that $\{T_{n,m}\}_m$ being a sequence of continuous real function on $l(p)$, for each n, then $\{T_n\}$ is also a sequence of continuous real function on $l(p)$ and $sup_n T_n(x) < \infty$. Then the result follows by arguing (as in) with uniform boundedness principle.

Sufficiency: We consider the case $1 < p_k < \infty$. Suppose that the conditions (i) and (ii) hold and $x \in l(p)$ since we know the following inequality(see Lascarides and Maddox) if $x, y \in \mathbb{C}$ and $N > 1$ then,

$$|xy| \le N(|x|^{q_k} N^{\frac{1}{q_k}} + |y|^{p_k}) \,,\ \left(p_k > 1, \frac{1}{p_k} + \frac{1}{q_k} = 1)\right)$$

We have $\left|\psi_{m,n}(Ax)\right| \le \sum_k (N|t(n,k,m)|^{q_k} N^{1/q} + |x_k|^{p_k})$

Hence $A \in (l(p), f_\infty)$.

Theorem 1.3

Prove that $A \in (l_\infty(p), V_\sigma^\infty)$ if and only if

$$sup_{n,m} \sum_k |t(n,k,m)| M^{\frac{1}{pk}} < \infty \ for\ all\ M > 1\ and\ n\,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (4)$$

Proof

Sufficient condition: Let equation (4) hold and $x \in l_\infty(p)$.

Then we have $\left|t_{m,n}(Ax)\right|^{pk} \le \sum_k |t(n,k,m)|^{pk} |x_k|^{pk}$

$$\le (|\textstyle\sum_k t(n,k,m)|^{pk} sup_k |x_k|^{pk})$$

$$\le (\textstyle\sum_k t(n,k,m)) M^{\frac{1}{pk}}$$

Now taking supremum over m,n both side we get $Ax \in V_\sigma^\infty \ for\ x \in l_\infty(p)$

That is $A \in (l_\infty(p), V_\sigma^\infty)$

Necessary condition: Let $A \in (l_\infty(p), V_\sigma^\infty)$ and we write $q_n(x) = sup_m |t(m, n\, A(x)|$. It is clear to see that $n \ge 0, q_n$ is continuous seminorm on $l_\infty(p)$ and q_n is pointwise bounded on $l_\infty(p)$.

Suppose (4) is not true. Then there exists $x \in l_\infty(p)\, with\ sup_n q_n(x) = \infty$. By the principal of condensation of singularities [5], the set $\left\{x \in l_\infty(p) : sup_n q_n(x)\ M^{\frac{1}{pk}} = \infty\right\}$ is the second category in l_∞ and hence nonempty, that is $x \in l_\infty(p)$ with $sup_n q_n(x) = \infty$. But this contradicts the facts that q_n is pointwise bounded on $l_\infty(p)$. Now by the Banach-Steinhauss theorem there is N such that $q_n(x) \le N\|x\|_1$ .... (5)

Now we define a sequence $x = x_k$ see[6] by

$$x_k = \begin{cases} sign\, t(n,k,m), 1 \le k \le k_0 \\ \quad 0 \;\; for \;\; k > k_0 \end{cases}$$

Then $x \in l_\infty(p)$. Applying this sequence to equation (5) we get equation (4). This complete the proof of the theorem.

VII. Conclusion

It is necessary to note that sequences are of significant importance in the field of mathematics and by extension , to sciences and beyond. The sequential arrangement of the functionary parts of some machines including constant applications in some aspect of human endeavours, such as the sequential order of the genetic made-up in all living organisms whose alteration can result into deformity (technically called mutations in biology) and others, were some of the motivating factors that necessitated the study of various sequences. In this paper, deliberate attempt was made to construct new sequence spaces so as to study some properties of the defined spaces. Further , we also determine the necessary and sufficient conditions on the matrix sequence $A = (A_n)$.

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